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Deterministic N-Qubit GHZ State Discrimination with Controlled-Not Operations

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Abstract We propose a scheme to deterministically realize N-qubit GHZ state measurement and discrimination. The scheme is simple and efficient in the senses that only controlled-not operations and single-qubit measurement are needed.

Keywords N-qubit GHZ state · Controlled-not operations

1 Introduction

Historically, the notion of entanglement was introduced by Schrödinger in 1935 [1], long before the dawn of the relatively young field of quantum information. Nowadays, entanglement has been served as a useful (in some cases unreplaceable) resource in quantum information processing and quantum computing [2–5]. So, as a necessity, understanding and employing entangled state become more and more important. Besides well-understood bipartite entangled states, there also exist multipartite entangled ones that, though less-understood, play a very significant role in quantum networking. Two inequivalent representatives of multipartite entangled states are the GHZ [6] and the W [7] states which cannot be converted to each other by local unitary operations and classical communication. Many schemes were proposed [8–10] and demonstrated in various context [11–16] to produce entanglement. In addition to perfect entanglement, the practical implementation of a concrete QIP and QC require successful measurement on certain numbers of qubits, such as two-qubit Bell-state measurement, N-qubit ($N \geq 3$) GHZ-state measurement (GSM). However, it was shown that complete Bell-state analysis using linear optics is not possible [17, 18], and that the optimal probability of success is only 50% [18–20], not to speak of the complete GHZ-state analysis. In the literatures, there exist very few scheme for realization of GSM [21]. In this paper, we propose a simple but effective method to realize arbitrary N-qubit ($N \geq 3$) GSM with the $N - 1$ controlled-not (CNOT) operations and N single-qubit measurements. For a concrete GSM implementation, we first show the effect of the standard GSM, i.e., the direct

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projection onto the N-qubit complete orthonormal basis, and then present the result of our alternative GSM method. Through comparison of the effects, we can show our GSM method are correct, i.e. the process of our scheme is indeed an implementation of GSM.

2 GHZ State Discrimination Scheme

We first show a simple three-qubit GSM for a single qubit plus two independent qubits of two Bell states. The qubits 1, 2 and 3, 4 are prepared in the same Bell state as

$$|\mathcal{B}\rangle = \frac{1}{\sqrt{2}}\sum_{i=0}^{1}|i, i\oplus 1\rangle, \tag{1}$$

and a single qubit 5 is in the form as

$$|\phi_5\rangle = \sum_{k=0}^{1} c_k|k\rangle. \tag{2}$$

The state of the total system can be written as

$$\begin{aligned}|\Psi_s\rangle &= \frac{1}{2}\sum_{i,j,k=0}^{1}|i, i\oplus 1\rangle_{12}|j, j\oplus 1\rangle_{34}c_k|k\rangle_5 \\ &= \frac{1}{2}\sum_{i,j,k=0}^{1}c_k|i, j, k\rangle_{135}|i\oplus 1, j\oplus 1\rangle_{24}.\end{aligned} \tag{3}$$

The complete orthonormal basis of the three qubits system has eight states which can be presented as

$$|G_{l,m,n}\rangle = \frac{1}{\sqrt{2}}\sum_{p=0}^{1}(-1)^{lp}|p, p\oplus m, p\oplus n\rangle, \tag{4}$$

with $l, m, n \in \{0, 1\}$ and $\oplus$ an addition mod 2. The inverse transformation is

$$|l, m, n\rangle = \frac{1}{\sqrt{2}}\sum_{p=0}^{1}(-1)^{lp}|G_{p,l\oplus m,l\oplus n}\rangle. \tag{5}$$

By virtue of (5), we can rewrite (3), in the form

$$\begin{aligned}|\Psi_s\rangle &= \frac{1}{2\sqrt{2}}\sum_{i,j,k,p=0}^{1}c_k(-1)^{ip}|G_{p,i\oplus j,i\oplus k}\rangle_{135}|i\oplus 1, j\oplus 1\rangle_{24} \\ &= \frac{1}{2\sqrt{2}}\sum_{k,p,m,n=0}^{1}c_k(-1)^{(n\oplus k)p}|G_{p,m,n}\rangle_{135}|n\oplus k\oplus 1, m\oplus n\oplus k\oplus 1\rangle_{24}.\end{aligned} \tag{6}$$

It is obvious from (6), if we makes standard GSM on qubits 1, 3 and 5 with outcomes $\{p, m, n\}$ corresponding to finding $|G_{p,m,n}\rangle_{135}$, the states 2 and 4 will be transformed into

$$|\psi\rangle_{24} = \sum_{k=0}^{1}(-1)^{(n\oplus k\oplus 1)p} c_k |n \oplus k, m \oplus n \oplus k\rangle_{24}. \tag{7}$$

In the following, we present an alternative method to implement the GSM, which need only two-qubit controlled-not (CNOT) operation and single qubit detection without using three-qubit interaction and GHZ states discrimination. We first perform two CNOT operations with the qubit 5 as the control and the qubit 1 and 3 as the targets respectively. As a result, the total state (3) of qubits 1, 2, 3, 4 and 5 is transformed into

$$\begin{aligned} |\Psi_s\rangle &= \frac{1}{2}\sum_{i,j,k=0}^{1} c_k |k \oplus i, k \oplus j, k\rangle_{135} |i \oplus 1, j \oplus 1\rangle_{24} \\ &= \frac{1}{2}\sum_{m',n',k=0}^{1} c_k \big|m', n', k\big\rangle_{135} \big|m' \oplus k \oplus 1, n' \oplus k \oplus 1\big\rangle_{24}. \end{aligned} \tag{8}$$

The two eigenstates of a qubit in the z-basis $\{|0\rangle, |1\rangle\}$ are related to those in the x-basis $\{|\tilde{0}\rangle, |\tilde{1}\rangle\}$ as

$$|l\rangle = \frac{1}{\sqrt{2}}\sum_{s=0}^{1}(-1)^{ls}|\tilde{s}\rangle, \tag{9}$$

$$|\tilde{s}\rangle = \frac{1}{\sqrt{2}}\sum_{l=0}^{1}(-1)^{sl}|l\rangle. \tag{10}$$

By virtue of (9), we can represent (8) in the z-basis of qubits 1, 3 and x-basis of qubit 5 as

$$|\Psi_s\rangle = \frac{1}{2\sqrt{2}}\sum_{m',n',k,s=0}^{1} c_k \big|m', n'\big\rangle_{13} (-1)^{(k\oplus 1)s} |\tilde{s}\rangle_5 \big|m' \oplus k, n' \oplus k\big\rangle_{24}. \tag{11}$$

It can be seen from (11) that if we make single-qubit measurements on the qubits 1 and 3 respectively in the z-basis and on the qubit 5 in the x-basis with outcomes $\{m', n', s\}$ corresponding to finding $|m', n', \tilde{s}\rangle_{135}$, then the state of qubits 2 and 4 is projected onto

$$|\psi\rangle_{24} = \sum_{k=0}^{1}(-1)^{(k\oplus 1)s} c_k \big|m' \oplus k, n' \oplus k\big\rangle_{24}. \tag{12}$$

Though two forms of the final state of the qubits 2 and 4, i.e. Equations (7) and (12) are different for the two process, they are essentially the same as each other. That is to say our method without using three-qubit interaction is the same as the standard GSM and the single-qubit measurement results of our method have the one-to-one correspondence with the eight GHZ states (see Table 1).

Next, we show the local GSM for the N qubits of N Bell states. For simplicity, we first show the case of $N = 3$. Suppose there are three identical Bell states $|\mathcal{B}\rangle_{12}$, $|\mathcal{B}\rangle_{34}$ and $|\mathcal{B}\rangle_{56}$ which are prepared in the form (1). It is known that if ones perform a GSM on the qubits

Table 1 Explicit expressions

$\vert m', n', \tilde{s}\rangle$	$\vert 0,0,\tilde{0}\rangle$	$\vert 0,0,\tilde{1}\rangle$	$\vert 0,1,\tilde{0}\rangle$	$\vert 0,1,\tilde{1}\rangle$	$\vert 1,0,\tilde{0}\rangle$	$\vert 1,0,\tilde{1}\rangle$	$\vert 1,1,\tilde{0}\rangle$	$\vert 1,1,\tilde{1}\rangle$
$G_{p,m,n}$	$G_{0,0,0}$	$G_{1,0,0}$	$G_{0,1,0}$	$G_{1,1,0}$	$G_{0,1,1}$	$G_{1,1,1}$	$G_{0,0,1}$	$G_{1,0,1}$

1, 3 and 5, the qubits 2, 4, and 6 will be projected onto a GHZ state due to entanglement swapping. Here, we present a similar GSM method as described in Sect. 2 which can also realize the same effect as the standard GSM to project the previously independent qubits to a GHZ state. In the following we first give the standard GSM process and then present our alternative GSM method. The total state of the six qubits system is

$$\begin{aligned}|\Psi_t\rangle &= |\mathcal{B}\rangle_{12} \otimes |\mathcal{B}\rangle_{34} \otimes |\mathcal{B}\rangle_{56}\\ &= \frac{1}{2\sqrt{2}} \sum_{i,j,k=0}^{1} |i, i \oplus 1\rangle_{12} \, |j, j \oplus 1\rangle_{34} \, |k, k \oplus 1\rangle_{56}\\ &= \frac{1}{2\sqrt{2}} \sum_{i,j,k=0}^{1} |i, j, k\rangle_{135} \, |i \oplus 1, j \oplus 1, k \oplus 1\rangle_{246}. \qquad (13)\end{aligned}$$

By virtue of (5), we can rewrite (13), in the form

$$\begin{aligned}|\Psi_t\rangle &= \frac{1}{4} \sum_{i,j,k,p=0}^{1} (-1)^{ip} \left|G_{p,i\oplus j,i\oplus k}\right\rangle_{135} |i \oplus 1, j \oplus 1, k \oplus 1\rangle_{246}\\ &= \frac{1}{4} \sum_{k,p,m,n=0}^{1} (-1)^{(n\oplus k)p} \left|G_{p,m,n}\right\rangle_{135} |n \oplus k \oplus 1, m \oplus n \oplus k \oplus 1, k \oplus 1\rangle_{246}. \qquad (14)\end{aligned}$$

It is obvious from (14), if we makes standard GSM on qubits 1, 3 and 5 with outcomes $\{p, m, n\}$ corresponding to finding $|G_{p,m,n}\rangle_{135}$, the states 2, 4 and 6 will be transformed into

$$|\psi\rangle_{246} = \frac{1}{\sqrt{2}} \sum_{k=0}^{1} (-1)^{(n\oplus k\oplus 1)p|n\oplus k,m\oplus n\oplus k,k\rangle_{246}}. \qquad (15)$$

In the following, we show how our alternative GSM method works. First, we makes two CNOT operations with the qubit 1 as the control and qubits 3 and 5 as the targets respectively. Here, the qubits 2 or 3 can also be as the control for the symmetry. The total state of the six qubits system will be transformed into

$$\begin{aligned}|\Psi_t\rangle &= \frac{1}{2\sqrt{2}} \sum_{i,j,k=0}^{1} |i, i \oplus j, i \oplus k\rangle_{135} \, |i \oplus 1, j \oplus 1, k \oplus 1\rangle_{246}\\ &= \frac{1}{2\sqrt{2}} \sum_{m,n,k=0}^{1} |n \oplus k, m, n\rangle_{135} \, |n \oplus k \oplus 1, m \oplus n \oplus k \oplus 1, k \oplus 1\rangle_{246}\\ &= \frac{1}{2\sqrt{2}} \sum_{m,n,l=0}^{1} |l, m, n\rangle_{135} |l \oplus 1, m \oplus l \oplus 1, n \oplus l \oplus 1\rangle_{246}. \qquad (16)\end{aligned}$$

Table 2 Explicit expressions

$\|\tilde{s}, m, n\rangle$	$\|\tilde{0}, 0, 0\rangle$	$\|\tilde{0}, 0, 1\rangle$	$\|\tilde{0}, 1, 0\rangle$	$\|\tilde{0}, 1, 1\rangle$	$\|\tilde{1}, 0, 0\rangle$	$\|\tilde{1}, 0, 1\rangle$	$\|\tilde{1}, 1, 0\rangle$	$\|\tilde{1}, 1, 1\rangle$
$\|G_{p,m,n}\rangle$	$\|G_{0,0,0}\rangle$	$\|G_{0,0,1}\rangle$	$\|G_{0,1,0}\rangle$	$\|G_{0,1,1}\rangle$	$\|G_{1,0,0}\rangle$	$\|G_{1,0,1}\rangle$	$\|G_{1,1,0}\rangle$	$\|G_{1,1,1}\rangle$

By virtue of (9), we can rewrite (16), in the form

$$|\Psi_t\rangle = \frac{1}{4} \sum_{m,n,l,s=0}^{1} (-1)^{ls} |\tilde{s}, m, n\rangle_{135} |l \oplus 1, m \oplus l \oplus 1, n \oplus l \oplus 1\rangle_{246}. \tag{17}$$

From (17) we can see that if we make single-qubit measurements on the qubits 3 and 5 in the z-basis respectively and on the qubit 1 in the x-basis with outcomes $\{s, m, n\}$ corresponding to finding $|\tilde{s}, m, n\rangle_{135}$, then the state of qubits 2, 4 and 6 is projected onto

$$|\psi\rangle_{246} = \frac{1}{\sqrt{2}} \sum_{l=0}^{1} (-1)^{(l\oplus 1)s} |l, m \oplus l, n \oplus l\rangle_{246}. \tag{18}$$

With simple calculations to (15) and (18) ones can verify that the final collapsed state of the qubits 2, 4 and 6 is identical for the two different GSM process. That is our GSM process with CNOT operations and single-qubit detection is the same as the standard GSM. The explicit correlations between the single-qubit measurement results $|\tilde{s}, m, n\rangle_{135}$ and the GHZ state measurement results $|G_{p,m,n}\rangle_{135}$ are shown in Table 2. The above three-qubit GSM can be generalized to the N-qubit case. Suppose there are N identical Bell states $|\mathcal{B}\rangle_{12}$,$|\mathcal{B}\rangle_{34}, \ldots, |\mathcal{B}\rangle_{(2N-1)2N}$ which are prepared in the form (1). The total state of the $2N$ qubits system can be written as

$$\begin{aligned}
|\Gamma\rangle &= |\mathcal{B}\rangle_{12} \otimes |\mathcal{B}\rangle_{34} \otimes \cdots \otimes |\mathcal{B}\rangle_{(2N-1)2N} \\
&= \frac{1}{(\sqrt{2})^N} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{2N}=0}^{1} |\alpha_1, \alpha_1 \oplus 1\rangle_{12} |\alpha_2, \alpha_2 \oplus 1\rangle_{34} \cdots |\alpha_{2N-1}, \alpha_{2N-1} \oplus 1\rangle_{(2N-1)2N} \\
&= \frac{1}{(\sqrt{2})^N} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{2N}=0}^{1} |\alpha_1, \alpha_2, \ldots, \alpha_{2N-1}\rangle_{1,3,\ldots,(2N-1)} \\
&\quad \times |\alpha_1 \oplus 1, \alpha_2 \oplus 1, \ldots, \alpha_{2N-1} \oplus 1\rangle_{2,4,\ldots,2N}.
\end{aligned} \tag{19}$$

We make $N-1$ CNOT operations with the qubit 1 (the same for the qubits $3, 5, \ldots, 2N-1$ for the symmetry) as the control and qubits $3, 5, \ldots, 2N-1$ as the targets respectively. The total state (19) of the six qubits system will be transformed into

$$\begin{aligned}
|\Gamma\rangle &= \frac{1}{(\sqrt{2})^N} \sum_{\alpha_1\alpha_2,\ldots,\alpha_{2N}=0}^{1} |\alpha_1, \alpha_1 \oplus \alpha_2, \ldots, \alpha_1 \oplus \alpha_{2N-1}\rangle_{1,3,\ldots,(2N-1)} \\
&\quad \times |\alpha_1 \oplus 1, \alpha_2 \oplus 1, \ldots, \alpha_{2N-1} \oplus 1\rangle_{2,4,\ldots,2N} \\
&= \frac{1}{(\sqrt{2})^N} \sum_{\alpha_1\beta_2,\ldots,\beta_{2N}=0}^{1} |\alpha_1, \beta_2, \ldots, \beta_{2N-1}\rangle_{1,3,\ldots,(2N-1)} \\
&\quad \times |\alpha_1 \oplus 1, \alpha_1 \oplus \beta_2 \oplus 1, \ldots, \alpha_1 \oplus \beta_{2N-1} \oplus 1\rangle_{2,4,\ldots,2N}.
\end{aligned} \tag{20}$$

We rewrite (20) in the x-basis of the qubit 1 as

$$|\Gamma\rangle = \frac{1}{(\sqrt{2})^N} \sum_{s\alpha_1\beta_2\ldots\beta_{2N}=0}^{1} (-1)^{\alpha_1 s} |\tilde{s}, \beta_2, \ldots, \beta_{2N-1}\rangle_{1,3,\ldots(2N-1)} \times |\alpha_1 \oplus 1, \alpha_1 \oplus \beta_2 \oplus 1, \ldots, \alpha_1 \oplus \beta_{2N-1} \oplus 1\rangle_{2,4,\ldots 2N}. \quad (21)$$

To complete the GSM ones can measure the qubit 1 in the x-basis and qubits $3, 5, \ldots, 2N-1$ in the z-basis respectively with outcomes $\{s, \beta_2, \ldots, \beta_{2N-1}\}$ corresponding to finding $|\tilde{s}, \beta_2, \ldots, \beta_{2N-1}\rangle_{1,3,\ldots,(2N-1)}$, then the state of qubits $2, 4, \ldots, 2N$ is projected onto

$$|\psi\rangle_{24\ldots(2N)} = \frac{1}{\sqrt{2}} \sum_{\alpha_1=0}^{1} (-1)^{(\alpha_l \oplus 1)s} |\alpha_1, \alpha_1 \oplus \beta_2, \ldots, \alpha_1 \oplus \beta_{2N-1}\rangle_{2,4,\ldots,l2N}. \quad (22)$$

3 Conclusion

In conclusion, we propose a simple but effective method to realize arbitrary N-qubit ($N \geq 3$) GSM with the $N-1$ controlled-not operations and N single-qubit measurements. For a concrete GSM implementation, we first show the effect of the standard GSM, i.e., the direct projection onto the N-qubit complete orthonormal basis, and then present the result of our alternative GSM method. Through comparison of the effects, we have shown our GSM method is correct. Our scheme can be implemented in experiment since the controlled-not gate can be realized in different context, such as the gates with single trapped cold ions have been experimentally demonstrated [22]. It is known GHZ-like state is crucial for many application purposes (see, e.g., [23–25]) and can already be generated in the laboratory [26], thus the study for this state are important and necessary.

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